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Singular configurations of structural systems

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Abstract

A singular configuration of a structural system is characterized by rank deficiency of the equilibrium matrix and kinematic matrix (the rank is lower than both the number of degrees of freedom and the number of constraints). Such configurations exist only in systems that are not geometrically invariant (underconstrained structural systems). Most interesting among them are systems with infinitesimal mobility which attracted attention of many prominent researchers. This paper puts the entire issue in a different perspective by addressing a critical, yet so far unexplored, aspect of singular configurations—their realizability. It turns out that the only generic, physically realizable type of a singular configuration is a system with first-order infinitesimal mobility, and even this cannot be constructed without inducing prestress of finite magnitude. All other singular configurations (unprestressable first-order mechanisms; higher-order mechanisms; and singular configurations of finite mechanisms) are unrealizable. Moreover, short of exact or symbolic calculation, they are also noncomputable and are just formal analytical constructs. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

Structural systems that are not geometrically invariant (underconstrained systems) generally do not possess a unique geometric configuration; they are kinematically mobile and allow finite displacements without any deformations of structural members. However, underconstrained systems admit certain special configurations where they lose kinematic mobility and can be prestressed. Such exceptional systems are well studied and widely used in engineering (tensegrity structures, large-span cable and membrane roofs and other prestressed and rapidly deployable structural systems for both terrestrial and space applications). The definitive property of these underconstrained systems is uniqueness of geometric configuration, i.e., kinematic immobility; yet, they possess a kind of elastic mobility whereby first-order infinitesimal displacements are possible at the expense of second or higher-order elongations in structural members.

Analytical statics and kinematics of systems with infinitesimal mobility originated in the works of Maxwell (1890), Mohr (1885) and Levi-Civita (1930). However, the subsequent progress in their investigation was slow, and it was not before the second half of this century that these systems have become the object of a new wave of research and the subject of a large body of technical

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publications. The main motivation has been the emergence of high-strength materials from which these systems benefit the most. One of the first comprehensive studies marking the resurgence of interest in these systems can be found in the volume edited by Rabinovich (1962).

Complete information on the kinematic properties of a system with ideal (undeformable, inextensible) positional constraints is contained in a set of simultaneous constraint equations:

$$F^i(X_1, \ldots, X_n, \ldots, X_N; C_i) = 0, \quad i = 1, 2, \ldots, C. \tag{1}$$

Here the C constraint functions F^i relate the N generalized coordinates, X_n, to the known geometric parameters, C_i, of the system (linear and angular sizes of the structural members). At least one solution to the constraint equations, $X_n = X_n^0$, is assumed to be known and is taken as the reference geometric configuration. Further investigation requires expanding the functions F^i into power series at the solution point $X_n = X_n^0$:

$$F^i_n x_n + (1/2!)F^i_{mn} x_m x_n + \cdots = 0, \quad m, n = 1, 2, \ldots, N. \tag{2}$$

Here x_n are infinitesimal increments of the respective coordinates (that is, virtual displacements of the system) and a repeated index denotes summation over the indicated range.

The linear terms of the expansion appear in the linearized constraint equations

$$F^i_n x_n = 0, \tag{3}$$

where the first derivatives,

$$F^i_n = \partial F^i / \partial X_{n|0}, \tag{4}$$

are the elements of the constraint function Jacobian matrix at X_n^0. The Jacobian matrix rank being $r = N$ is a necessary and sufficient analytical criterion of a geometrically stable (invariant) system; in this case all virtual displacements and, the more so, all finite kinematic displacements are zero. At $r < N$ the system is underconstrained and eqn (3) can be solved in terms of properly selected $V = (N - r)$ virtual displacements chosen as independent; each of them defines a linearly independent virtual displacement mode.

The existence of V nontrivial solutions to the linearized constraint eqn (3), representing V virtual displacement modes, indicates V-th degree of virtual indeterminacy of the system. Virtual indeterminacy and the associated virtual mobility is necessary for, and almost always entails, kinematic indeterminacy, i.e., the possibility of kinematic displacements. However, since the latter are determined by the nonlinear eqns (1) or (2), it may still happen that the given solution $X_n = X_n^0$ is an isolated point in the configuration space (if not the entire space). Then the system does not allow any displacements, and its virtual mobility, although a principal property, is purely formal; the system is kinematically immobile and has a unique geometric configuration. Such exceptional underconstrained systems, allowing only virtual displacements but no kinematic ones, are said to possess infinitesimal mobility and are called infinitesimal mechanisms.

Both the applied analysis of systems with infinitesimal mobility and the exploration of their basic properties were expedited by the use of computers and modern matrix tools. In the beginning, only first-order infinitesimal mechanisms with a single degree of indeterminacy were investigated (Tarnai, 1980; Pellegrino and Calladine, 1986). Later, more-complex systems, those with higher-order infinitesimal mobility and higher degree of indeterminacy, were addressed by Calladine and

Pellegrino (1991) and by Kuznetsov (1991). The elastic stability and vibration issues for infinitesimal mechanisms were explored recently by Volokh and Vilnay (1997).

The definitive property of an infinitesimal mechanism is that its infinitesimal first-order displacements are possible at the expense of second- or higher-order elongations of structural members. From this perspective, infinitesimal mechanisms are classified depending on the lowest order of elongations required to produce first-order displacements. Tarnai (1989) formulated the most elaborate formal definition of the order of an infinitesimal mechanism and compared it to several known alternatives. His definition has been challenged by Connelly and Servatius (1994) who, however, acknowledge that their own approach, based on the discrete-geometric concept of order of rigidity, leads to paradoxical conclusions.

A general method for evaluating the order of infinitesimal mobility has been suggested by Koiter (1989). Although the problem is purely geometric, the method takes advantage of the well developed nonlinear theory of elastic stability, utilizing the notion of elasticity and strain energy. Nevertheless, despite a considerable effort (Salerno, 1992), practical evaluation of the order of infinitesimal mobility for particular systems still presents a formidable problem.

This paper puts the entire issue in a different perspective by addressing a critical, yet so far unexplored, aspect of systems with infinitesimal mobility—their physical realizability.

2. Singular configurations: generic vs nongeneric

An analytical definition of a singular geometric configuration of a structural system is rank deficiency of the Jacobian (and equilibrium) matrix:

$$N > r < C. \tag{5}$$

A given geometric configuration of a structural system is described by two kinds of variables—endogenous, X_n (coordinates uniquely locating all structural members and material points); and exogenous, C_i (linear and angular sizes of the structural members, support spacing, etc.). Taking exogenous variables C_i as control parameters, allows some basic concepts from nonlinear dynamics and the theory of singularities to be applied to the analysis of singular configurations of structural systems, in particular, infinitesimal mechanisms.

In kinematical terms, an infinitesimal mechanism is defined as a system that possesses virtual mobility but no kinematic mobility—its geometric configuration is unique. Such exceptional configurations occur as a result of degeneracy and evolve from two opposite directions. Varying the control parameters of a geometrically invariant system (Jacobian matrix rank $r = N \leqslant C$) may produce a singular configuration in spite of a sufficient number of constraints, C. Such systems are called quasi-invariant; the simplest example is a von Mises truss (Fig. 1a) with three collinear pins

Fig. 1. Ordinary and singular configurations of geometrically invariant system: (a) ordinary, invariant; (b) singular, quasi-invariant.

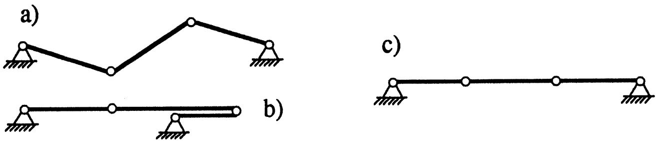

Fig. 2. Ordinary and singular configurations of geometrically variant system: (a) ordinary, variant; (b) singular, variant; (c) singular, quasi-variant.

(Fig. 1b). On the other hand, manipulating the control parameters of a geometrically variant system (Fig. 2a and b) may lead to a loss of kinematic mobility, producing a quasi-variant system (Fig. 2c). Note that, according to the accepted terminology, kinematic immobility and the resulting uniqueness of geometric configuration are not synonymous with geometric invariance; both quasi-invariant and quasi-variant systems are kinematically immobile but, according to (5), admit virtual displacements, hence, are not geometrically invariant.

Thus, of the four kinematic types of structural systems, two are generic and two degenerate, singular. Generic types are determined solely by their structural topology (the system connectivity) whereas the system geometry is, generally, irrelevant; varying the control parameters changes the geometric configuration of the system, but almost always (short of degeneration) leaves its kinematic type intact. In contrast, singular types are determined by the system geometry, specified by the entire set of endogenous and exogenous variables. Upon exiting from the singular configuration, a degenerate system reverts to the generic type of its origin, either geometrically invariant or variant. The following discussion is focused on investigating various geometric singularities, along with the properties of the corresponding singular configurations and the resulting implications for the physical realization.

To begin with, the basic classification of the kinematic types of structural systems as generic and degenerate (singular) types is not thorough. The reason is that singularities, in turn, also fall into two categories—generic and nongeneric (Arnold, 1984). This categorization, by taking a closer look at singular configurations, paves the way to both refining and extending the above basic kinematic classification. The distinction between generic and nongeneric singularities stems from the underlying concept of structural stability (a mathematical concept unrelated to structures).

The fact is that the exact values of control parameters can never be known in a real situation. Therefore, a basic requirement of any physically meaningful model must be that minute changes in the values of control parameters, as a rule, do not produce any abrupt, "essential", change in the system. (An example of an opposite, exceptional, situation is the proverbial last straw breaking the camel's back.) Models satisfying the above requirement are called structurally stable; only such models can be meaningful, observable and realizable in actual non-transient phenomena and systems. A mathematical formulation of the concept reads:

A system of equations (a model) is structurally stable if any sufficiently small change in the control parameters does not result in an "essential" change in the solutions of the system.

The immediate difficulty with this idea is in defining an "essential" change. In nonlinear dynamics (e.g., Jackson, 1989), this is defined as a topological change in the phase portrait of the system; accordingly, *topological orbital equivalence* of the original and perturbed systems is taken as a criterion of structural stability. Looking for a suitable criterion of structural stability for the problem in hand, it seems logical to compare the feasible displacement modes of the original and perturbed geometric configurations. When choosing between virtual and kinematic displacements, recall that as long as the kinematic displacements are infinitesimal, they are coincident with the virtual displacements. However, whereas all underconstrained structural systems possess virtual mobility, only geometrically variant ones are kinematically mobile. Clearly, virtual displacements is the unavoidable choice when dealing with a strictly local, configuration-specific issue of structural stability.

The above reasoning leads to a notion of *virtual modal equivalence* (topological equivalence of virtual displacement modes) as a sought criterion for defining an "essential" change in the system behavior. Thus, the condition of structural stability is the equivalence of virtual displacement modes in the original and perturbed geometric configurations of a given system.

In this light, it becomes clear that the geometries of the two generic kinematic types (invariant and variant) are structurally stable. In fact, the structural stability is a formal expression of their generic nature and can be taken as a rigorous definition for the respective two types. (It is a sad semantic accident that a geometrically unstable, variant structural system—even a house of cards!—is, in conventional mathematical terms, structurally stable. Still, tinkering with the well established terminology seems inappropriate in this case.)

A more disturbing conclusion is that the two degenerate kinematic types (quasi-invariant and quasi-variant) are structurally unstable; hence, should not be physically realizable—whereas in fact, these systems not only exist but are widely and successfully used in engineering practice.

Before addressing this apparent paradox, it would be useful to show rigorously that any system in a degenerate geometric configuration is structurally unstable. This follows immediately from the analytical sign of degeneracy—rank deficiency of the Jacobian matrix. Recall that for any system, the degree of virtual indeterminacy, $V = (N-r)$, is the number of nontrivial solutions of linearized constraint eqn (3), i.e., the number of independent virtual displacement modes. Obviously, a drop in the rank, by increasing the number of such modes, rules out virtual modal equivalence between the degenerate geometric configuration and an adjacent ordinary, generic, configuration. Thus, degenerate geometry entails structural instability; in particular, this should be true for the two degenerate kinematic types of systems with infinitesimal mobility—quasi-invariant and quasi-variant.

As a simple illustration, consider once again the above pin-bar chains. For the geometrically invariant von Mises truss (Fig. 1a), acquiring a singular configuration with collinear pins is accompanied by an increase in the degree of virtual indeterminacy (and the number of virtual displacement modes) from $V = 0$ to $V = 1$. For the geometrically variant pin-bar chain (Fig. 2), two singular configurations are possible—one kinematically mobile (Fig. 2b), the other immobile (Fig. 2c); in both cases the degree of virtual indeterminacy increases from $V = 1$ to $V = 2$.

Tarnai (1988), noticing that geometric degeneration is accompanied by an increase in the number of displacement modes, called this phenomenon kinematic bifurcation. In view of this notion, it can now be said that structural instability of a singular geometric configuration has the form of virtual bifurcation (in finite mechanisms, virtual bifurcation is also kinematic bifurcation). With structural instability of all singular geometric configurations ascertained, the question of their existence is to be investigated.

3. Generic infinitesimal mechanisms

In terms of constraint eqn (1), the necessary and sufficient analytical criterion of a system with infinitesimal mobility (an infinitesimal mechanism) involves two requirements:

(1) the rank of the constraint Jacobian matrix in the reference configuration is $r < N$, and
(2) the given solution of constraint equations is an isolated point in the configuration space.

There are no general analytical criteria or procedures verifying whether or not the second condition is met. One particular, and only sufficient, criterion is based on the concept of prestressability: an underconstrained system allowing prestress (a stable state of self-stress—see below) is an infinitesimal mechanism.

As shown in analytical statics, the equilibrium matrix for a structural system in a given configuration is a transpose of the constraint Jacobian matrix. In the absence of external loading, the equilibrium equations in terms of constraint reactions (member forces), Λ_i, are

$$F^i_n \Lambda_i = 0. \tag{6}$$

A self-stress is a statically possible stress state given by a nontrivial solution to homogeneous eqn (6). According to (5), at least one such solution always exists for a system in a singular configuration, due to $r < C$; the number of linearly independent nontrivial solutions, $S = (C - r)$, is the degree of statical indeterminacy of the configuration. Note that a statically determinate configuration is nonsingular; it cannot be an infinitesimal mechanism and belongs to one of the two generic types—geometrically invariant (at $r = N$) or variant (at $r < N$).

It should be emphasized that self-stress is just a virtual, purely formal, stress state; only if this state is stable, an actual, physical state of prestress can exist in the system. Furthermore, it turns out that prestressability alone, i.e., just the possibility of prestress, rather than its presence, constitutes the above sufficient criterion of infinitesimal mobility. Indeed, prestressability is a statical manifestation of a purely geometric fact: it means that in the given geometric configuration, one of the control parameters, such as a bar length, attains a minimum subject to fixed magnitudes of the remaining control parameters.

The prestressability criterion provides a clue to explaining the paradoxical existence of infinitesimal mechanisms; the answer is found in the physical means of their construction. A structural system is manufactured or assembled with some finite precision reflecting various geometric imperfections. This, however, is the case only with systems of generic types, where a small imprecision in the control parameters is nothing but a small perturbation, inconsequential by virtue of structural stability of these systems. In contrast, structural instability of singular configurations, including infinitesimal mechanisms, means that their construction requires perfect geometric precision. It is prestressability or, more exactly, its underlying cause—the above mentioned conditional minimum property—that compensates for the lack of precision. Physically, the required singular configuration is attainable, say, by inserting a turnbuckle into one of the system bars and adjusting it to the extreme. Another method is to apply a set of nodal forces to a variant system and, upon reaching the equilibrium state, restrain the loaded nodes with external supports and remove the loads. In both cases the system follows a continuous path terminating at the boundary of the configuration space; this amounts to self-terminating control bringing the system to a singular and prestressable geometric configuration.

Still, a singular borderline configuration thus achieved may be deemed structurally stable only in the context of ideal, undeformable material and in the absence of any subsequent perturbations, e.g., thermal fluctuations or support settlements. Such a far reaching idealization makes structural stability of a prestressable, but unprestressed, ideal system a judgement call. However, since real materials are elastic, prestress of finite magnitude ensures structural stability. Indeed, elastic stability, more broadly defined as absence of "essential" changes in the system under small perturbations of all relevant parameters, becomes synonymous with structural stability within the same set of exogenous parameters. Thus, an elastic system in a prestressed singular configuration is structurally stable and its singularity is generic. Obviously, this stability is only local, confined to the finite amount of the elastic strain induced by prestress; some finite perturbations (say, a temperature change) still may produce "essential" changes in the virtual displacement modes.

In summary, the role of prestress is significant in several ways. First, as a nontrivial solution of homogeneous equilibrium equations, it indicates singularity of the equilibrium matrix, constraint Jacobian matrix, and of the given geometric configuration. Second, physical prestress overrides all geometric imperfections, including lack of precision in the member sizes and in the process of assembly; with the singularity engendered by statics, and not by the infeasible exact geometry, the resulting singular configuration is somewhat different from the nominal one. Third, the finite elastic strain induced by prestress makes the singularity locally structurally stable, i.e., generic.

It turns out that prestressed elastic systems constitute the only class of generically singular structural systems; all other singular configurations, both kinematically immobile and mobile, are nongeneric and, therefore, physically unrealizable. Furthermore, certain infinitesimal mechanisms for which the state of self-stress is known to be stable, cannot be actually prestressed in the given singular configuration, thus indicating that their singularity is also nongeneric.

Before reviewing various classes of nongeneric infinitesimal and finite mechanisms, it is necessary to consider yet another far reaching implication of the concept of structural stability. Aside from the already discussed issue of physical realizability, it leads to an associated notion of computability, known as the Fredkin postulate (Jackson, 1989):

"There is a one-to-one mapping between what is possible in the real world, and what is theoretically possible in the digital simulation world", and the corollary "that which cannot, in principle, be simulated on a computer, cannot be part of physics".

The link with structural stability is clear: the unavoidably finite precision of computing and, especially, of input data (such as bar lengths), in effect, amounts to small perturbations of control parameters. For a structurally stable system this does not result in an "essential" change in the solutions, making a meaningful computing feasible. Indeed, identifying a singular configuration calls for evaluating the Jacobian matrix rank, which can be done, for example, using the singular value decomposition. Based on the feasible precision of the input data and computing, an error tolerance is established and the singular values above it are counted. The effective rank thus obtained (Strang, 1988) is an acceptable compromise in the case of generic singularity, i.e., for prestressed underconstrained elastic systems, as discussed above. Accordingly, these, and only these, generically singular, configurations are both physically realizable and computable, whereas singular nongeneric configurations are unrealizable and, generally, noncomputable.

An interesting situation arises in exceptional cases of ostensibly exact computations, made feasible by idealizations or overall simplicity of the system (not surprisingly, only such exact cases, mostly with integer numerical parameters or symbolic calculations, are encountered in the technical literature). Here the obtained results and conclusions are based on a tacit assumption of zero manufacturing and computing tolerances, an unacceptable assumption for singular nongeneric configurations. The implications of this observation for higher-order infinitesimal mechanisms and singular configurations of finite mechanisms are rather drastic.

4. Higher-order and finite mechanisms

The most important attribute of a singular configuration is its control space codimension, which is the reduction in the number of independent control parameters relative to an adjacent nonsingular configuration. This reduction, due to additional relations imposed on the control parameters in order to obtain singularity, equals the number of the additional relations. Codimension is the key to the following systematic review and assessment of various known classes of singular configurations.

To begin with, certain first-order infinitesimal mechanisms are known to be unprestressable (Kuznetsov, 1991). Such a system (a compound mechanism) is an assembly of two or more nonoverlapping singular subsystems. Two examples of compound mechanisms, one quasi-invariant, the other quasi-variant, are shown in Fig. 3. Each mechanism is comprised of two singular subsystems and has two independent states of self-stress. However, since no combination of the two states is stable, neither mechanism is prestressable. As a result, each mechanism, involving two structurally unstable singular subsystems of codimension 1, is “doubly” unrealizable; a slight departure from the nominal geometry (say, due to imperfect bar lengths) makes the system in Fig. 3a geometrically invariant and the one in Fig. 3b, variant.

Even-order mechanisms constitute another known broad class of unprestressable infinitesimal mechanisms. Consider an underconstrained system in Fig. 4 as an assembly of a three-bar and two-bar subsystems joined by a pin at the middle of the inclined bar. With a rectilinear two-bar chain, the system configuration is singular. The path P of the pin, as prescribed by the three-bar subsystem, is an asymmetric curve with a horizontal tangent at the origin. The boundary path of the same pin in the two-bar subsystem is a circular arc with the same horizontal tangent and the center at the support. The divergence between the two paths can be expressed as a polynomial function of the tangential displacement of the pin; this constraint discontinuity polynomial is a convenient measure of the order of infinitesimal mobility. For the configurations in Fig. 4a–c, the two paths share a tangent, hence, the linear term in the discontinuity polynomial is absent, the constraints are compatible to at least the first order, and the system is not geometrically invariant.

If the middle support (Fig. 4a) is above the local center of curvature, O, of the path P, the two-bar chain has a minimum length compatible with the given lengths of all the remaining bars in the system. This configuration is prestressable, the constraints are compatible only to the first order,

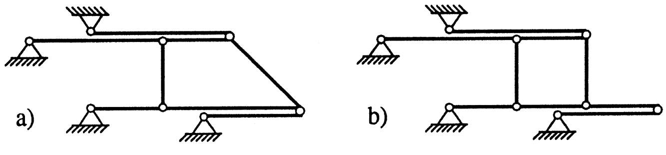

Fig. 3. Unprestressable, nongeneric first-order infinitesimal mechanisms: (a) quasi-invariant; (b) quasi-variant.

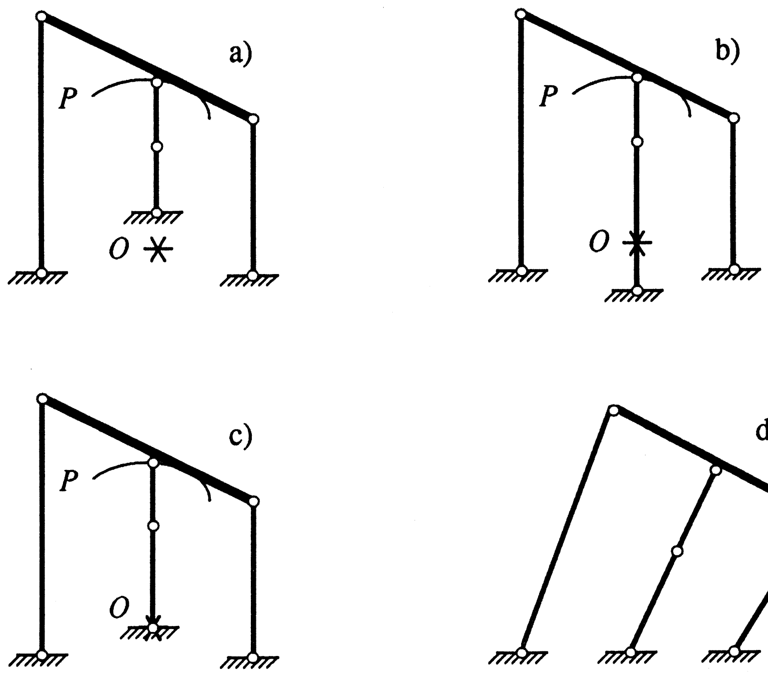

Fig. 4. Transformations of nongeneric second-order mechanism: (a) generic prestressed first-order mechanism; (b) unrealizable singular configuration of finite mechanism; (c) nongeneric, unrealizable second-order mechanism; (d) generic prestressed first-order mechanism.

and the system is a first-order infinitesimal mechanism. It is structurally stable and its singularity is generic, insensitive to small imperfections. For example, although the two-bar chain must be rectilinear, it may not be exactly parallel to the two lateral bars, as long as all three intersect at one point. If the support is below point O (Fig. 4b), the two-bar chain will no longer be rectilinear, which entails kinematic mobility within the domain of constraint compatibility. The system is still in a singular configuration, but its statically possible self-stress is unstable. The singularity is nongeneric, making the configuration unrealizable; if constructed, the system is a finite mechanism in some nonsingular, generic configuration in the vicinity of the singular one.

Requiring that the support is exactly at point O (Fig. 4c), imposes an additional relation on the control parameters (bar lengths), thus reducing the number of independent ones and thereby increasing the singularity codimension from 1 to 2. Now both the first- and second-order terms are absent in the discontinuity polynomial, constraints are compatible to the second order, and the system is a second-order infinitesimal mechanism. It is, however, unprestressable, therefore, nongeneric and unrealizable. Indeed, the prerequisite for prestressability—the conditional minimum property of one of the control parameters—is absent in this, as well as in all other, even-order mechanisms. Instead, only a stationary value is attained, which leads to a drop in the Jacobian matrix rank, hence, a singular geometry, but does not entail a minimum. If constructed, the system reverts to one of its two generic alternatives. Note that, regardless of the initial geometry, shortening one of the two middle bars eventually reduces its length to a minimum compatible with

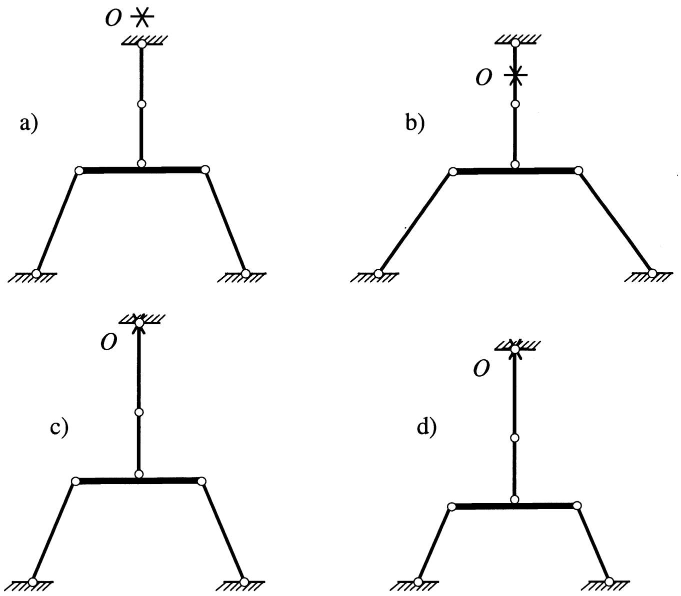

Fig. 5. Singularity codimension and higher-order infinitesimal mobility: (a) generic first-order mechanism (codimension 1); (b) unrealizable singular configuration of finite mechanism (codimension 1); (c) unrealizable third-order mechanism (codimension 3); (d) unrealizable fifth-order mechanism (codimension 5).

all other bar lengths. As a result, the system acquires a prestressable singular configuration and is a first-order infinitesimal mechanism (Fig. 4d). In it, the two-bar chain is rectilinear, with its extension passing through the intersection point of the lateral bar extensions.

As shown in the earlier discussion, even in a prestressable configuration, an underconstrained system is structurally unstable unless prestress actually exists. Surprisingly, prestressability (the statical possibility of a stable state of self-stress) does not yet guarantee that prestress can be actually implemented; it turns out that the state of prestress, although elastically stable, still may be unrealizable in the given geometric configuration.

The type of an underconstrained system in Fig. 5 depends on the location of the upper support relative to the intersection point, O, of the two lateral bar extensions. If the support is below O (Fig. 5a), the system is a prestressable, first-order infinitesimal mechanism; if above (Fig. 5b), it is a singular nongeneric configuration of a finite mechanism. In both cases the singularity codimension is one. Adjusting the lateral bar slopes such that the intersection point is at the support (Fig. 5c), produces a higher-order mechanism; indeed, the two paths of the connecting pin, prescribed by the two subsystems, are locally compatible to within at least two orders of smallness. In contrast to the system in Fig. 4, all odd-order terms in the discontinuity polynomial vanish due to presumed symmetry. Accordingly, the mechanism in Fig. 5c is of the third order, with the singularity codimension 3. Next, leaving the lateral bar slopes intact, their lengths can be adjusted such as to annul the fourth-order term in the discontinuity polynomial. The result (with symmetry in mind) is singularity of codimension 5 and a fifth-order infinitesimal mechanism (Fig. 5d). In both cases, the condition of a constrained minimum is met, assuring prestressability.

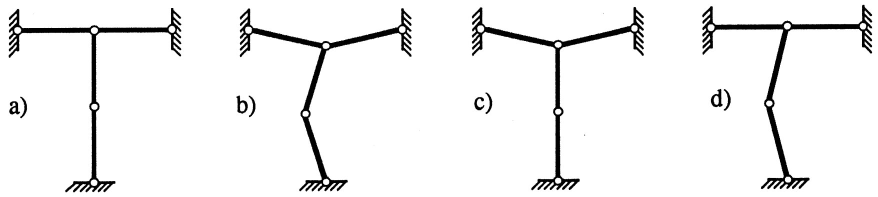

Fig. 6. Transformations of partially prestressable, nongeneric third-order mechanism: (a) unrealizable third-order mechanism; (b) generic type—geometrically invariant; (c) and (d) two versions of prestressed, generic first-order mechanism.

The conclusions reached on higher-order infinitesimal mobility of the mechanisms in Fig. 5c and d are rigorously verifiable (assuming exact computation) by any of the known alternative approaches, including the nonlinear theory of elastic stability. Thus, the presented singular systems are prestressable infinitesimal mechanisms, and it can be expected that actually inducing prestress should, as usual, stabilize them just like the mechanism in Fig. 5b.

Yet, higher-order mechanisms are elusive. Just as with any singular configuration, those in Fig. 5c and d are structurally unstable and require perfect precision in the control parameters (bar lengths and support locations). Under any perturbation, such as a geometric imperfection, or even an attempt of inducing prestress, either system will revert to one of its two generic incarnations—nonsingular, geometrically variant or, if in fact prestressed, singular, quasi-variant. Remarkably, the generic singular configuration—a prestressed quasi-variant system—is a first-order infinitesimal mechanism. The ostensible third- and fifth-order mechanism turn out to be just formal artifacts, and the same is true with their fundamental geometric property—prestressability as a higher-order infinitesimal mechanism. The state of prestress, in spite of being elastically stable, is unrealizable for singular configurations of codimension greater than one.

Similar behavior is observed in partially prestressable higher-order mechanisms. For example, the classic third-order infinitesimal mechanism (Fig. 6a) is a codimension 3 singular configuration that is nongeneric and unrealizable; with a minute perturbation, the system reverts to one of its alternative, structurally stable, generic configurations—either nonsingular, geometrically invariant (Fig. 6b) or, if prestressed, singular, quasi-invariant (Fig. 6c and d).

To sum up, the role of prestress depends on the system configuration. For prestressable first-order mechanisms, prestress is the means (and the only means!) for realization of the singular geometric configuration by making it generic. For higher-order mechanisms, a state of prestress, in spite of its elastic stability, is unrealizable in the given configuration; more precisely, the configuration is unrealizable, since prestress does not render it generic. The reason, as observed in the foregoing examples, is that each unit increase in the order of infinitesimal mobility requires imposing an additional relation on the control parameters, i.e., an increase in the singularity codimension. This higher codimension precludes both physical realizability of higher-order mechanisms (even prestressable ones) and their computability as well. Indeed, for codimension 1, the computational compromise of using the effective matrix rank, needed to recognize the singularity, has been overcome by prestress; the latter made the singularity generic within the context of elasticity, i.e., for an expanded state space. There are no known physical means for remedying higher-codimensional singularity of higher-order mechanisms. That is why, short of exact comput-

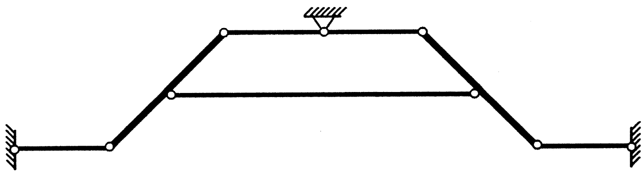

Fig. 7. Cusp mechanism of Connelly and Servatius.

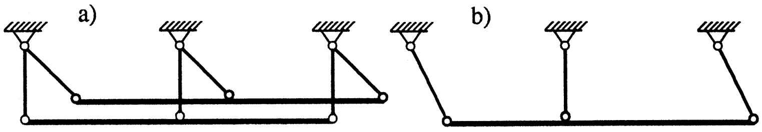

Fig. 8. Globally statically indeterminate finite mechanism: (a) unrealizable singular configuration of finite mechanisms; (b) generic type—geometrically invariant.

ing, any attempt to evaluate the order of infinitesimal mechanisms is futile as a matter of principle. Moreover, as has been mentioned above, even ostensibly exact computations tacitly assume a perfect geometry, an assumption unacceptable for nongeneric singular configurations.

Going over to kinematically mobile systems (finite mechanisms) note, first of all, that their singular configurations are, generally, unprestressable, structurally unstable, hence, nongeneric and unrealizable. For example, with slightly imperfect bar lengths, the singular configuration in Fig. 2b is unfeasible: the three bars, generally, cannot be collinear. As long as there are no means for a perfect length control, a pin-bar mechanism allowing for such a configuration cannot be constructed.

A more sophisticated and much more interesting example is a cusp mechanism (Fig. 7) composed by Connelly and Servatius (1994) and claimed in their Proposition 2 to be "third order rigid but not rigid" (using their given definition of rigidity). In fact, from the supporting calculations (which are algebraic) it follows that this is a singular, codimension 3, configuration of a finite mechanism. This configuration is unprestressable, structurally unstable, nongeneric and unrealizable, figuratively speaking, to the third degree (according to the singularity codimension).

One peculiar kind of singular geometry is that of a globally statically indeterminate finite mechanism (Kuznetsov, 1991). As seen from a simple example in Fig. 8a, this is a degenerate configuration of a geometrically invariant system; with any imperfection (say, not all of the three support bars are parallel) the system reverts to its generic type—invariant (Fig. 8b). Still, although formally invariant, the system, regardless of its material, has a very low elastic stiffness. For all practical purposes, such a system behaves as a finite mechanism with elastic interference: very large displacements are possible at the expense of small elastic strains (the source of the interference). Interestingly, shortening one of the support bars brings the system into a generic singular configuration where the three support bar extensions intersect at one remote point (cf the system in Fig. 4d). Even though the system is formally a prestressed first-order infinitesimal mechanism, its global behavior is still that of a finite mechanism with elastic interference.

5. Conclusions

(1) The only generic, physically realizable type of a singular configuration of a structural system is a system with first-order infinitesimal mobility, either quasi-invariant or quasi-variant. Implementation of these systems requires prestressing.
(2) All other singular configurations of structural systems are nongeneric, hence, unrealizable and, generally, noncomputable (except for exact or symbolic calculation). These configurations are just formal analytical constructs; their list includes all of the following:
   (a) unprestressable infinitesimal mechanisms, in particular, all even-order mechanisms;
   (b) higher-than-first order infinitesimal mechanisms, including prestressable ones;
   (c) singular configurations of finite mechanisms (without a load).
(3) Short of exact computing, any attempt to evaluate the order (if higher than first) of an infinitesimal mechanism is futile.

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